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ON NAGUMO'S EQUATION OF A NEURON

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1. Introduction

In this lecture we shall consider an equation of the type

$$\partial^2 u/\partial t^2 = \partial^3 u/\partial t\partial x^2 + f(u)\,\partial u/\partial t + g(u). \tag{1}$$

This equation is a general form of the equation

$$\partial^2 u/\partial t^2 = \partial^3 u/\partial t\partial x^2 - \mu(1-u+\varepsilon u^2)\partial u/\partial t - u,$$
$$\mu > 0, \quad \varepsilon > 0,$$

which was proposed by J. Nagumo(1962) as an equation of a neuron.

R. Arima and Y. Hasegawa(1963) have proved that there exists a unique solution of (1) with the initial and boundary conditions:

$$\begin{aligned} u(x,0) &= u_0(x), \quad \partial u/\partial t(x,0) = u_1(x) \text{ for } x \geqq 0, \\ u(0,t) &= \psi(t) \quad \text{for } t \geqq 0, \end{aligned} \tag{2}$$

under suitable conditions.

Here, we shall consider the equation (1) with the conditions:

$$\begin{aligned} u(x,0) &= \varphi(x) \in C^2[0,1], \quad \partial u/\partial t(x,0) = \psi(x) \in C^4[0,1], \\ u(0,t) &= a(t) \in C^3[0,\infty), \quad u(1,t) = b(t) \in C^3[0,\infty). \end{aligned} \tag{3}$$

Now, we put the following assumptions on f, g:

$$\begin{aligned} & f,\ g \in C^2(-\infty,\infty), \\ & -F(u^2+1) \leqq f(u) \leqq F, \\ & |g(u)| \leqq G|u|. \end{aligned} \tag{4}$$

The compatibility conditions are the following:

$$a(0) = \varphi(0), \quad b(0) = \varphi(1),$$

$$(5)\qquad \begin{aligned} a'(0) &= \psi(0), \quad b'(0) = \psi(1), \\ a''(0) &= \psi''(0) + f(a(0))\,a'(0) + g(a(0)), \\ b''(0) &= \psi''(1) + f(b(0))\,b'(0) + g(b(0)). \end{aligned}$$

Under the assumptions (3),(4),(5), we have the following result. Let T be any positive number.

Theorem. There exists a solution $u(t,x) \in C^2([0,T];C^2[0,1])$ satisfying (1) and (3).

By using the fundamental solution of heat equation R. Arima and Y. Hasegawa have obtained the integro-differential equation associated with (1),(2) and obtained apriori estimates. They have constructed a local solution of the integro-differential equation by the method of successive approximations and obtained a global solution by using apriori estimates. Our existence proof of a local solution is achieved by replacing $\partial u/\partial t$ by a difference scheme and by a compactness argument. In the construction of a local solution we don't use any particular properties of heat equation. However, we could'nt simply obtain global apriori estimates, unless the boundary conditions are homogeneous. Hence, in order to obtain a global solution we use an energy form for our problem, which is similar to that given by R. Arima and Y.Hasegawa for the problem (1), (2).

2. A construction of a solution

We transform the equation (1) to a system of equations

$$(6)\qquad \begin{aligned} \partial v/\partial t &= v'' + f(u)\,v + g(u), \\ \partial u/\partial t &= v. \end{aligned}$$

Let N be a positive integer sufficiently large and let us divide the interval $[0,T_0]$ $(T_0 \leqq T)$ by points $t_n = nh$ $(h = T_0/2^N,\ n = 0,1,\dots,2^N)$. We define functions $\{u_n\}$ $(n = -1,0,\dots,2^N)$, $\{v_n\}$ $(n = 0,1,\dots,2^N)$

inductively as follows:

$$u_0 = \varphi,$$
$$v_0 = \psi - \left\{(1-x)\frac{h}{2}a''(0) + x\frac{h}{2}b''(0)\right\},$$
$$\frac{v_n - v_{n-1}}{h} = v_n'' + f(u_{n-1})v_n + g(u_{n-1}),$$
(7) $$\frac{u_n - u_{n-1}}{h} = v_n,$$
$$u_n(0) = a(t_n), \quad u_n(1) = b(t_n),$$
$$u_{-1} = u_0 - h v_0,$$
$$v_{-1} = v_0 - h v_0'' - h f(u_{-1})v_0 - h g(u_0).$$

Further, we put

(8) $$z_0 = \frac{v_0 - v_{-1}}{h} = v_0'' + f(u_{-1})v_0 + g(u_{-1})$$

and for each n $(n = 1,2,\ldots,2^N)$

(8') $$z_n = \frac{v_n - v_{n-1}}{h}, \quad w_n = \frac{z_n - z_{n-1}}{h}.$$

In the Banach space $X = C[0,1]$ with the supremum norm

$$\|u\| = \sup_{0\le x\le 1} |u(x)|,$$

we construct Cauchy polygons by

$$P_N(t) = \frac{(t-t_{n-1})v_n + (t_n - t)v_{n-1}}{h},$$
$$Q_N(t) = \frac{(t-t_{n-1})u_n + (t_n - t)u_{n-1}}{h} \quad \text{for } t\in[t_{n-1}, t_n]$$
$$(1 \le n \le 2^N).$$

Suppose, for a moment, that there exists a compact set $K \subset C[0,1]$ (independent of N) such that

$$\{u_n\}, \{v_n\}, \{z_n\} \subset K,$$

and $\{w_n\}$ is equi-bounded (independent of N). Then, noticing that

$$d^+P_N(t)/dt = z_n,$$
$$d^+Q_N(t)/dt = v_n \quad \text{for } t \in [t_{n-1}, t_n),$$

we can conclude that the sequences $\{P_N(t)\}$, $\{Q_N(t)\}$, $\{d^+P_N(t)/dt\}$ and $\{d^+Q_N(t)/dt\}$ are equi-continuous in t and hence are normal families in X. By the construction (7) we have that

$$d^+P_N(t)/dt = P_N''(t_n) + f(Q_N(t_{n-1}))\,P_N(t_n) + g(Q_N(t_{n-1})),$$
$$d^+Q_N(t)/dt = P_N(t_n) \qquad \text{for } t \in [t_{n-1}, t_n),$$
$$P_N(0) = \psi - \{(1-x)\frac{h}{2}a''(0) + x\frac{h}{2}b''(0)\},$$
$$Q_N(0) = \varphi.$$

Noticing the closedness of the Laplace operator and d/dt, we have by the continuity of f and g that

$$dP(t)/dt = P''(t) + f(Q(t))\,P(t) + g(Q(t)),$$
$$dQ(t)/dt = P(t) \qquad \text{for } t \in [0, T],$$
$$P(0) = \psi, \quad Q(0) = \varphi,$$

where

$$P(t) = \lim_{N\to\infty} P_N(t), \quad Q(t) = \lim_{N\to\infty} Q_N(t).$$

By the construction of u_n we have that $P(t)(\cdot) \in C[0,1]$ satisfies

$$P(t)(0) = a(t), \quad P(t)(1) = b(t) \quad \text{for } t \geq 0.$$

Consequently, we shall construct $\{u_n\}$, $\{v_n\}$, which satisfy (7) and prove the equi-boundedness of $\{u_n\}, \{v_n\}, \{z_n\}, \{w_n\}$, which implies the existence of such a compact set K as stated above.

If we could find $\{w_n\}$, we can find $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ as follows:

$$z_n = z_{n-1} + h w_n = z_{n-2} + h w_{n-1} + h w_n$$
$$= z_0 + h(w_1 + w_2 + \cdots + w_n),$$
$$v_n = v_0 + h(z_1 + z_2 + \cdots + z_n),$$
$$u_n = u_0 + h(v_1 + v_2 + \cdots + v_n).$$

The equation that w_n $(n \geq 2)$ should satisfy is the following:

$$
\begin{aligned}
h^2 w_n'' &= h(w_n - w_{n-1}) - h^2 f(u_{n-1})\,w_n \\
&\quad - [f(u_{n-1}) - 2f(u_{n-2}) + f(u_{n-3})]\,v_{n-1} \\
&\quad - h[f(u_{n-1}) - f(u_{n-3})]\,z_{n-1} - [g(u_{n-1}) - 2g(u_{n-2}) + g(u_{n-3})],
\end{aligned} \tag{9}
$$

$$w_n(0) = \frac{a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3}}{h^3},$$

$$w_n(1) = \frac{b_n - 3b_{n-1} + 3b_{n-2} - b_{n-3}}{h^3}.$$

The equation that w_1 should satisfy is the following:

$$(10)\quad \begin{aligned} h^2 w_1'' &= h w_1 - f(u_0)(v_1 - v_0) + (f(u_{-1}) - f(u_0))v_0 \\ &\quad - (g(u_0) - g(u_{-1})) - h z_0'', \end{aligned}$$

$$w_1(0) = \frac{z_1(0) - z_0(0)}{h}, \quad w_1(1) = \frac{z_1(1) - z_0(1)}{h}.$$

In order to find a solution of (9) and (10), we shall construct majorant and mimorant functions for (9) and (10). By the compatibility condition (4) and the smoothness condition (3) of f, g there can be shown that there exists a positive constant M_0 such that

$$|u_n(0)|, |u_n(1)|, |v_n(0)|, |v_n(1)|, |z_n(0)|, |z_n(1)|,$$
$$|w_n(0)|, |w_n(1)| \leqq M_0 \quad (n \geqq 1), \quad \|u_0\|, \|v_0\| \leqq M_0.$$

The functions $\overline{\omega}_1(x) = M_0 + \gamma_1$, $\underline{\omega}_1(x) = -M_0 - \gamma_1$ and $\overline{\omega}_n(x) = M_0 + \gamma_n$, $\underline{\omega}_n(x) = -M_0 - \gamma_n$,

$$\gamma_1 = \max\{0,\ \|f\|_{M_0}\|z_1\| + \|f'\|_{M_0}\|v_0\|^2 + \|g'\|_{M_0}\|v_0\| + \|z_0''\| - K\},$$

$$(11)\quad \begin{aligned} (1 - h\|f(u_{n-1})\|)\gamma_n &= \gamma_{n-1} + \frac{1}{h}\|f(u_{n-1}) - 2f(u_{n-2}) + f(u_{n-3})\|\,\|v_{n-1}\| \\ &\quad + \|f(u_{n-1}) - f(u_{n-3})\|\,\|z_{n-1}\| + \frac{1}{h}\|g(u_{n-1}) - 2g(u_{n-2}) + g(u_{n-3})\| \end{aligned}$$

can be shown to be majorant, minorant functions for (10) and (9), respectively. Hence, there exists $\{w_n\}$ $(n \geqq 1)$, each of which satisfies (9),(10).

Next, we shall show that $\{u_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{w_n\}$ are equi-bounded. If we put

$$\overline{v_n} = v_n - \{(1-x)v_n(0) + x v_n(1)\} \quad (1 \leqq n \leqq 2^N),$$

$\overline{v_n}$ satisfies

$$\begin{aligned} \overline{v_n}'' - \frac{1}{h}\overline{v_n} &= -\frac{1}{h}\overline{v_{n-1}} - f(u_{n-1})\overline{v_n} \\ &\quad - \{(1-x) f(u_{n-1}) v_n(0) + x f(u_{n-1}) v_n(1)\} \end{aligned}$$

$$-g(u_{n-1}) - \{(1-x)z_n(0) + x z_n(1)\},$$

$$\overline{v_n}(0) = \overline{v_n}(1) = 0.$$

From the property of the Laplace operator we have

(12) $\|\overline{v_n}\| \leqq \|\overline{v_{n-1}}\| + hF\|\overline{v_n}\| + hG\|u_{n-1}\| + hM_0F(\|u_{n-1}\|^2+1) + hM_0.$

From

$$u_n = u_{n-1} + h\{\overline{v_n} + (1-x)v_n(0) + x v_n(1)\},$$

we have for $1 \leqq n \leqq 2^N$

(13) $\|u_n\| \leqq \|u_{n-1}\| + h(\|\overline{v_n}\| + M_0).$

Determine successively α_n and β_n by

$$\alpha_0 = \|\varphi\|,$$
$$\beta_0 = \|\psi\| + 1,$$
$$\beta_n = \beta_{n-1} + hF\beta_n + hG\alpha_{n-1} + hM_0F(\alpha_{n-1}^2+1) + hM_0$$
$$\alpha_n = \alpha_{n-1} + h(\beta_n + M_0).$$

Then the inequalities (12) and (13) imply the estimates

$$\|u_n\| \leqq \alpha_n, \quad \|\overline{v_n}\| \leqq \beta_n \quad \text{for } 0 \leqq n \leqq 2^N,$$

which can be verified by the induction with respect to n. Denote by $\alpha_N(t)$ $\beta_N(t)$ polygonal functions with their vertices (nh, α_n), (nh, β_n). If we have $N \to \infty$, $\{\alpha_N(t)\}$, $\{\beta_N(t)\}$ converge to continuous functions $\alpha(t)$, $\beta(t)$, which satisfy the following differential equations

$$d\alpha/dt = \beta + M_0,$$
$$d\beta/dt = F(\beta + \alpha^2 + 1) + G\alpha + M_0$$

with

$$\alpha(0) = \|\varphi\|, \qquad \beta(0) = \|\psi\| + 1.$$

If we put

$$\gamma(t) = \alpha(t)^2 + \beta(t)^2,$$

$\gamma(t)$ satisfies the following differential inequality

$$d\gamma/dt \leqq C(\gamma+1)^2$$

where C is a positive constant depending only on F, G, M_0. This inequality has a continuous solution on $[0,T_0]$, where

$$T_0 = \frac{1}{C\{1+\|\varphi\|^2+(\|\psi\|+1)^2\}},$$

which means the equi-boundedness of $\{\alpha_N(t)\}$, $\{\beta_N(t)\}$ and hence of $\{u_n\}$, $\{v_n\}$ The method of such estimates as stated just above is due to that by M. Hukuhara, which simplifies the estimation for the system of such inequalities as the type (12),(13). By noticing the equalities (8,8') and (11) we can similarly conclude that $\{z_n\}$, $\{w_n\}$ are equi-bounded(that is, $\|u_n\|, \|v_n\|, \|z_n\|, \|w_n\| \leqq M$). Consequently, the following estimates

$$\|v_n''\| \leqq \|z_n\| + \|f\|_M \|v_n\| + \|g\|_M \leqq M + M\|f\|_M + \|g\|_M \equiv M',$$

$$\|u_n''\| \leqq \|u_{n-1}''\| + h\|v_n''\| \leqq \|u_0''\| + TM',$$

$$\begin{aligned} h\|z_n''\| &= \|v_n'' - v_{n-1}''\| = \|z_n - f(u_{n-1})v_n - g(u_{n-1}) \\ &\qquad - z_{n-1} + f(u_{n-2})v_{n-1} + g(u_{n-2})\| \\ &\leqq h(\|w_n\| + \|f'\|_M \|v_{n-1}\|^2 + \|f\|_M\|z_n\| + \|g'\|_M \|v_{n-1}\|) \\ &\leqq h(M + M^2\|f'\|_M + M\|f\|_M + M\|g'\|_M) \end{aligned}$$

($\|f\|_M$, for example, means that $\|f\|_M = \sup_{\|u\| \leqq M} |f(u)|$.) imply that there exists a compact set $K \subset C[0,1]$ such that

$$\{u_n\}, \{v_n\}, \{z_n\} \subset K.$$

Hence, we can construct a solution of (1) and (3) on $[0,T_0]$. Similarly as R. Arima and Y. Hasegawa have done, we can construct an energy function to our problem and have a global solution.

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